A STRONG HYPERBOLICITY PROPERTY OF LOCALLY SYMMETRIC VARIETIES

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ABSTRACT. We show that all subvarieties of a quotient of a bounded symmetric domain by a sufficiently small arithmetic discrete group of automorphisms are of general type. This result corresponds through the Green-Griffiths-Lang's conjecture to a well-known result of Nadel.

1. INTRODUCTION

1.1. **Main result.** An arithmetic locally symmetric variety is by definition a complex analytic space which is isomorphic to a quotient of a bounded symmetric domain $\mathcal{D}$ by an arithmetic lattice $\Gamma \subset \mathrm{Aut}(\mathcal{D})$, see section 5.1 for a reminder. By a theorem of Baily-Borel [BB66], every arithmetic locally symmetric variety admits a canonical structure of algebraic variety. In this situation, Tai [AMRT75] and Mumford [Mum77] have shown that there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that the algebraic variety $\Gamma'\backslash\mathcal{D}$ is of general type. Recall that an irreducible smooth projective complex variety X is said of general type if it has enough pluricanonical forms to make the canonical rational maps $X \dashrightarrow \mathbb{P}(\mathrm{H}^0(X, \omega_X^{\otimes m})^\vee)$ birational onto their images for $m \gg 1$. In that case, every smooth projective complex variety birational to X is of general type too. An irreducible complex algebraic variety X, not necessarily projective or smooth, is then said of general type if any smooth projective complex variety birational to X is of general type.

The following theorem, which is our main result in this paper, strengthens considerably the result of Tai and Mumford.

Theorem 1.1 (Main result)**.** *Let $\mathcal{D}$ be a bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ an arithmetic lattice. Then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that all subvarieties of $\Gamma'\backslash\mathcal{D}$ are of general type.*

Remarks 1.2. (i) Theorem 1.1 follows from a stronger statement: it follows from Theorem 5.3 that there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that any smooth projective variety birational to a subvariety of $\Gamma'\backslash\mathcal{D}$ has a big cotangent bundle (in the sense that the tautological line bundle on the corresponding projective bundle is big, see Definition 2.6). But a smooth projective variety with a big cotangent bundle is of general type by the work of Campana, Peternell and Păun, cf. Theorem 2.9.
(ii) Note that in general it is necessary to take a finite index subgroup. Already for $\mathcal{D} = \Delta$ the unit disk in $\mathbb{C}$ and $\Gamma(n) := \ker(\mathrm{Sl}(2,\mathbb{Z}) \to \mathrm{Sl}(2,\mathbb{Z}/n\mathbb{Z}))$, it is well-known and easy to check that the level-n modular curve $Y(n) := \Gamma(n)\backslash\Delta$ is of general type exactly when $n > 6$.

Date: June 8, 2020.

(iii) If $\mathcal{D}$ is a bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ an arithmetic lattice, then by the results of Tai and Mumford already mentioned there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that $\Gamma' \backslash \mathcal{D}$ is of general type. Moreover, it follows from the main result of [Nad89] (see Theorem 1.9 below) that Γ' can be chosen so that all subvarieties *of dimension 1* of $\Gamma' \backslash \mathcal{D}$ are of general type. Our proof of Theorem 1.1 gives a new approach to these results that works also for subvarieties of intermediate dimensions.
(iv) When Γ is a cocompact lattice, Theorem 1.1 is a consequence of [BKT13, Theorem 3.1]. In this case, any torsion-free subgroup $\Gamma' \subset \Gamma$ of finite index satisfies the conclusion.

As a direct application of Borel's algebraization theorem [Bor72, Theorem 3.10], one obtains:

Corollary 1.3. *Let $\mathcal{D}$ be a bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ an arithmetic lattice. Then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that any projective complex variety which admits a non-empty Zariski open subset with an immersive holomorphic map to $\Gamma' \backslash \mathcal{D}$ is of general type.*

Note that by [BKT13, Theorem 3.1] a projective complex variety endowed with a (*everywhere defined*) generically immersive holomorphic map to a quotient of a bounded symmetric domain by a torsion-free discrete group Γ of automorphisms is of general type. More generally, if we assume that the variety is only quasi-projective, then it has to be of log-general type by [Bru18, Theorem 0.2]. Corollary 1.3 refines greatly this last result when Γ is an *arithmetic lattice*. Let us emphasize here that the strategy leading to the proof of Theorem 1.1 is different from the approach of [BKT13] and [Bru18]. In particular, this strategy can be used to give a new proof of [BKT13, Theorem 3.1] and [Bru18, Theorem 0.2].

Given two positive intergers g and n, let $\mathcal{A}_g(n)$ denote the moduli stack of principally polarized abelian varieties of dimension g with a level-n structure. The corresponding coarse moduli space $A_g(n)$ is an arithmetic locally symmetric variety whose associated bounded symmetric domain $\mathcal{D}$ is the Harish-Chandra's realization (cf. [Mok89, Theorem 1, p.94]) of the Siegel half-space of rank g. In this special case we prove the more precise statement:

Theorem 1.4. *For any $g \geq 1$ and any $n > 12 \cdot g$, every subvariety of $A_g(n)$ is of general type.*

It follows immediately from Theorem 1.4 that a smooth complete complex algebraic variety which admits a non-empty Zariski-open subset parameterizing a family of principally polarized abelian varieties of dimension g with a level-n structure and whose corresponding period map is generically immersive is of general type as soon as $n > 12 \cdot g$ (we will show more precisely that its cotangent bundle is big, cf. Theorem 4.3).

Remarks 1.5. (i) There is an important literature about the Kodaira dimension of the $A_g(n)$ and their subvarieties. In particular, it is known that for $g \geq 7$ the coarse moduli space A_g (and a fortiori every $A_g(n)$ for $n \geq 1$) is of general type. This was first proved for g divisible by 24 by Freitag [Fre77] and then improved to $g \geq 9$ by Tai [Tai82], $g \geq 8$ by Freitag [Fre83] and finally $g \geq 7$ by Mumford [Mum83]. On the other hand, A_g is known to be unirational for $g \leq 5$ (due to [Don84] for

$g = 5$, [Cle83] for $g = 4$, classical for $g \leq 3$). Later, Weissauer [Wei86] proved that for $g \geq 13$ every subvariety of the coarse moduli space A_g of codimension $\leq g - 13$ is of general type. A fortiori, the same result holds for all $A_g(n)$ with $g \geq 13$ and $n \geq 1$. However, A_g always contains the codimension $g - 1$ subvariety of negative Kodaira dimension $A_1 \times A_{g-1}$ (being uniruled). On the other hand, it follows from [Nad89, Rou16] that any curve in $A_g(n)$ is of general type when $n > 6 \cdot g$.
(ii) As this paper was being written, Abramovich and Várilly-Alvarado posted a preprint [AVA18] where they prove (cf. Theorem 1.4 in op. cit.) that for any closed subvariety X in $\mathcal{A}_g$, there exists a level n_X (a priori depending on X) such that the irreducible components of the preimage of X in $\mathcal{A}_g(n)$ are of general type for $n > n_X$. Our result gives an explicit n_X which works for all X.

1.2. **The geometric Lang conjecture for the Baily-Borel compactification of arithmetic locally symmetric varieties.** Given an arithmetic locally symmetric variety $\Gamma\backslash\mathcal{D}$, we denote by $\overline{\Gamma\backslash\mathcal{D}}^*$ its Satake-Baily-Borel compactification. It is a normal projective variety which contains $\Gamma\backslash\mathcal{D}$ as a Zariski-open dense subset, and the boundary $\overline{\Gamma\backslash\mathcal{D}}^* - \Gamma\backslash\mathcal{D}$ is stratified by arithmetic locally symmetric varieties. For example, in the case of $A_g(n)$, the projective variety $\overline{A_g(n)}^*$ admits a natural stratification by locally closed subvarieties, each of which being canonically isomorphic to $A_{g'}(n)$ for some $0 \leq g' \leq g$ [FC90, Theorem 2.5, p.252]. Therefore, as a direct application of Theorem 1.4, one obtains:

Corollary 1.6. *For any $g \geq 1$ and any $n > 12 \cdot g$, every subvariety of $\overline{A_g(n)}^*$ is of general type.*

Similarly, as an application of Theorem 1.1 and the construction of the Baily-Borel compactification, one can show:

Corollary 1.7. *Let $\mathcal{D}$ be a bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ an arithmetic lattice. Then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that all subvarieties of $\overline{\Gamma'\backslash\mathcal{D}}^*$ are of general type.*

We would like now to explain how our results fits in the framework of Lang's conjectures. Recall that a complex analytic space X is called hyperbolic in the sense of Brody if there is no non-constant holomorphic map $\mathbb{C} \to X$. Given a projective complex variety X, one can measure the deviation from Brody-hyperbolicity of the corresponding analytic space by introducing its exceptional subvariety $\mathrm{Exc}(X) \subset X$, which is by definition the Zariski closure of the union of the images of all non-constant holomorphic maps $\mathbb{C} \to X$. A famous conjecture of Green-Griffiths and Lang predicts that this subset has an interpretation in the realm of algebraic geometry:

Conjecture 1.8 (Green-Griffiths, Lang, cf. [GG80, Lan86])**.** *Let X be an irreducible projective complex variety. Then X is of general type if and only if $\mathrm{Exc}(X) \neq X$.*

Observe that Conjecture 1.8 implies that given a projective complex variety X, the irreducible components of its exceptional locus $\mathrm{Exc}(X)$ are not of general type, and that any irreducible subvariety of X not contained in $\mathrm{Exc}(X)$ is of general type.

Conjecture 1.8 is known for some special classes of varieties, including subvarieties of abelian varieties, smooth projective surfaces with a big cotangent bundle and subvarieties of a generic hypersurface of high degree. See respectively

[Blo26, Och77, Kaw80], [Bog77, McQ98], [Pac04, Bro17] and the references therein.

The exceptional subvariety of compactifications of arithmetic locally symmetric varieties has been studied by Nadel, see also [Nog91, HT06, Rou16].

Theorem 1.9 (Nadel, [Nad89])**.** *Let $\mathcal{D}$ be a bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ an arithmetic lattice. Then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that the image $f(\mathbb{C})$ of any non-constant holomorphic map $f : \mathbb{C} \to \overline{\Gamma'\backslash\mathcal{D}}$, where $\overline{\Gamma'\backslash\mathcal{D}}$ is any compactification of $\Gamma'\backslash\mathcal{D}$, is contained in $\overline{\Gamma'\backslash\mathcal{D}} - \Gamma'\backslash\mathcal{D}$.*

In other words, the exceptional subvariety of any compactification $\overline{\Gamma'\backslash\mathcal{D}}$ is included in the boundary $\overline{\Gamma'\backslash\mathcal{D}} - \Gamma'\backslash\mathcal{D}$. Using the canonical stratification of the Baily-Borel compactification by arithmetic locally symmetric varieties, it follows that $\overline{\Gamma''\backslash\mathcal{D}}^*$ is Brody hyperbolic for some $\Gamma'' \subset \Gamma'$ of finite index. Corollary 1.7 is then the corresponding statement predicted by Conjecture 1.8.

1.3. **Conjectural arithmetic implications for Shimura varieties.** Our main result has some (partly conjectural) implications for Shimura varieties that we now briefly discuss. Let $(\mathbf{G}, X)$ be a Shimura datum: $\mathbf{G}$ is a reductive $\mathbb{Q}$-group and X a $\mathbf{G}(\mathbb{R})$-conjugation class of morphisms $\mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}_m \to \mathbf{G}_{\mathbb{R}}$ satisfying Deligne's axioms, cf. [Del71, Del79]. If K is a compact-open subgroup of $\mathbf{G}(\mathbb{A}_f)$, one has the corresponding Shimura variety:

$$\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} = \mathbf{G}(\mathbb{Q})\backslash(X \times \mathbf{G}(\mathbb{A}_f)/K),$$

which is a finite disjoint union of arithmetic locally symmetric varieties (note that the corresponding arithmetic groups are of congruence type). As before, its Baily-Borel compactification $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}^*$ is a normal projective complex variety which contains $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ as a Zariski-open dense subset, the boundary $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}^* - \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is stratified by Shimura subvarieties, and Corollary 1.7 becomes in this framework

Corollary 1.10. *Let $(\mathbf{G}, X)$ be a Shimura datum and K be a compact-open subgroup of $\mathbf{G}(\mathbb{A}_f)$. There exists a compact-open subgroup $K' \subset K \subset \mathbf{G}(\mathbb{A}_f)$ such that any subvarieties of $\overline{\mathrm{Sh}_{K'}(\mathbf{G}, X)_{\mathbb{C}}}^*$ is of general type.*

Following Bombieri and Lang, the exceptional subvariety should also have an arithmetic significance:

Conjecture 1.11 (Bombieri, Lang)**.** *If F is a subfield of $\mathbb{C}$ finitely generated over $\mathbb{Q}$ and X a projective variety defined over F, then the set of F-rational points of X lying outside of $\mathrm{Exc}(X_{\mathbb{C}})$ is finite.*

It is not difficult to verify that if Conjecture 1.8 is true, then the exceptional locus $\mathrm{Exc}(X)$ of a complex variety X is defined over any field of definition of X.

By the work of Shimura, Deligne, Milne, Shih and Borovoi, $Sh_K(\mathbf{G}, X)_{\mathbb{C}}$ admits a canonical model over the reflex field $E(\mathbf{G}, X)$ associated to the Shimura datum $(\mathbf{G}, X)$. Moreover, its Baily-Borel compactification $\overline{\mathrm{Sh}_K(\mathbf{G}, X)}^*$ is defined over the same field. Therefore, as a consequence of Bombieri-Lang conjecture, one obtains the following

Conjecture 1.12. *Let $(\mathbf{G}, X)$ be a Shimura datum and K be a compact-open subgroup of $\mathbf{G}(\mathbb{A}_f)$. There exists a compact-open subgroup $K' \subset K \subset \mathbf{G}(\mathbb{A}_f)$ such that for any finitely generated extension F of $E(\mathbf{G}, X)$, the set $\overline{\mathrm{Sh}_{K'}(\mathbf{G}, X)}^*(F)$ of F-points of $\overline{\mathrm{Sh}_{K'}(\mathbf{G}, X)}^*$ is finite.*

See [UY10] for a partial result in direction of this conjecture.

In the case of $\mathcal{A}_g$, in view of Corollary 1.6, the Bombieri-Lang conjecture would have the following consequence (compare with [AVA17, AVA18]):

Conjecture 1.13. *For any $g \geq 1$, any $n > 12 \cdot g$ and any field F finitely generated over $\mathbb{Q}$, there is only a finite number of isomorphism classes of principally polarized abelian varieties defined over F of dimension g with a level-n structure.*

1.4. **Organisation of the paper.** Our proof of Theorem 1.1 relies strongly on Hodge theory. We first prove in section 3 a criterion (Theorem 3.3) insuring that an algebraic variety supporting a variation of Hodge structure of a special form is of general type. Our proof of this criterion is reminiscent of a strategy introduced by Viehweg and Zuo in their study of hyperbolicity properties of moduli spaces of canonically polarized varieties [VZ02]. We then verify this criterion for Siegel modular varieties of high level in section 4 and for general arithmetic locally symmetric varieties in section 5. In the Siegel case, this is done by looking at the variation of Hodge structure coming from the middle degree relative cohomology of the universal family of abelian varieties parametrized by $\mathcal{A}_g$. In the general case, we use the canonical variation of Hodge structure of Calabi-Yau type constructed on any bounded symmetric domain by Gross [Gro94] and Sheng-Zuo [SZ10].
There are two competing notions of bigness for torsion-free coherent sheaves in the literature. Since we use both in this paper, we recall their definitions in section 2 and prove a result about them (Lemma 2.5) that we couldn't find in the literature.

1.5. **Notations.** In this paper, a smooth log pair (X, D) is a smooth complex algebraic variety X together with $D \subset X$ a union of smooth divisors crossing normally. A log pair (X, D) is said projective when X is projective. A morphism of log pairs $f : (X, D) \to (Y, E)$ is a morphism $f : X \to Y$ such that $f^{-1}(E) \subset D$. A (projective smooth) log-compactification of a smooth complex variety U is a projective smooth log pair (X, D) with an identification $X - D \simeq U$. In the sequel, all varieties will supposed to be irreducible.

2. Different notions of positivity for torsion-free sheaves

In this section, we recall for the reader convenience different positivity notions for torsion-free sheaves on smooth projective complex varieties that we will use later in this paper.

2.1. We begin with some notions due to Viehweg. For details and proofs, the reader is referred to [Vie83, Lemma 1.4] and [Vie95, p.59-67].

Definition 2.1. Let X be a complex quasi-projective scheme. A coherent sheaf $\mathcal{F}$ on X is globally generated at a point $x \in X$ if the natural map $\mathrm{H}^0(X, \mathcal{F}) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{F}$ is surjective at x.

Definition 2.2 (Viehweg)**.** Let X be a smooth projective complex variety and $\mathcal{F}$ a torsion-free sheaf on X. Let $i : V \hookrightarrow X$ denotes the inclusion of the biggest open subset on which $\mathcal{F}$ is locally free.

(1) We say that $\mathcal{F}$ is weakly positive over the dense open subset $U \subset V$ if for every ample invertible sheaf $\mathcal{H}$ on X and every positive integer $\alpha > 0$ there exists an integer $\beta > 0$ such that $\widehat{S}^{\alpha \cdot \beta}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}^{\beta}$ is globally generated over U.
(2) We say that $\mathcal{F}$ is Viehweg-big over the dense open subset $U \subset V$ if for any line bundle $\mathcal{H}$ there exists $\gamma > 0$ such that $\widehat{S}^{\gamma}\mathcal{F} \otimes \mathcal{H}^{-1}$ is weakly positive over U.

Here the notation $\widehat{S}^k\mathcal{F}$ stands for the reflexive hull of the sheaf $S^k\mathcal{F}$, i.e. $\widehat{S}^k\mathcal{F} = i_*(S^k i^*\mathcal{F})$.
We say that $\mathcal{F}$ is weakly positive (resp. Viehweg big) if there exists a dense open subset $U \subset V$ such that $\mathcal{F}$ is weakly positive (resp. Viehweg big) over U.

Remark 2.3. If $\mathcal{F}$ is locally free, then $\mathcal{F}$ is nef if and only if it is weakly positive over X, and $\mathcal{F}$ is ample if and only if it is Viehweg-big over X.

Lemma 2.4 (Viehweg)**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be torsion-free sheaves on a smooth projective complex variety X.*

(i) If $\mathcal{F} \to \mathcal{G}$ is a morphism, surjective over U, and if $\mathcal{F}$ is weakly positive over U, then $\mathcal{G}$ is weakly positive over U.
(ii) Let $f : Y \to X$ be a morphism between two smooth projective complex varieties. If $\mathcal{F}$ is weakly positive over $U \subset X$ and $f^{-1}(U)$ is dense in Y, then $f^*\mathcal{F}/(f^*\mathcal{F})_{tors}$ is weakly positive over $f^{-1}(U)$.
(iii) If $\mathcal{F} \to \mathcal{G}$ is a morphism, surjective over U, and if $\mathcal{F}$ is Viehweg-big over U, then $\mathcal{G}$ is Viehweg-big over U.
(iv) If $\mathcal{F}$ is weakly positive and $\mathcal{H}$ is a Viehweg-big line bundle, then $\mathcal{F} \otimes \mathcal{H}$ is Viehweg-big.
(v) Let $f : Y \to X$ be a morphism between two smooth projective complex varieties, which is finite over an open $V \subset X$. If $\mathcal{F}$ is Viehweg-big over U and $f^{-1}(U \cap V)$ is dense in Y, then $f^*\mathcal{F}/(f^*\mathcal{F})_{tors}$ is Viehweg-big over $f^{-1}(U \cap V)$.

2.2. We introduce now a weaker notion of bigness. Let $\mathcal{E}$ be a vector bundle on a smooth projective complex variety X. Let $\pi : \mathbb{P}(\mathcal{E}) := \mathrm{Proj}_{\mathcal{O}_X}(\mathrm{Sym}\,\mathcal{E}) \to X$ be the projective bundle of one-dimensional quotients of $\mathcal{E}$ and $\mathcal{O}_{\mathcal{E}}(1)$ be the tautological line bundle which fits in an exact sequence $\pi^*\mathcal{E} \to \mathcal{O}_{\mathcal{E}}(1) \to 0$.

Lemma 2.5. *The following assertions are equivalent:*

(1) The line bundle $\mathcal{O}_{\mathcal{E}}(1)$ is Viehweg-big.
(2) For some (resp. any) Viehweg-big line bundle $\mathcal{H}$, there exists an injective map $0 \to \mathcal{H} \to S^k\mathcal{E}$ for some $k > 0$.
(3) For some (resp. any) Viehweg-big torsion-free sheaf $\mathcal{F}$, there exists a non-zero map $\mathcal{F} \to S^k\mathcal{E}$ for some $k > 0$.

Definition 2.6. A vector bundle $\mathcal{E}$ on a smooth projective complex variety X is called big (in the sense of Hartshorne) if it satisfies the equivalent conditions of Lemma 2.5.

Lemma 2.5 implies that a Viehweg-big vector bundle is big, but the converse is not true (consider for example the rank 2 vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1$). Note however that the two notions coincide for line bundles.

Proof of Lemma 2.5. First observe that given a (non necessarily Viehweg-big) torsion-free sheaf $\mathcal{F}$ on X and locally-free sheaf $\mathcal{E}$ such that the line bundle $\mathcal{O}_{\mathcal{E}}(1)$ is Viehweg-big, there exists a non-zero map $\mathcal{F} \to S^k\mathcal{E}$ for any sufficiently big k. Indeed, it is sufficient to show the existence of a section of $\mathcal{O}_{\mathcal{E}}(k) \otimes \pi^*\mathcal{F}^\vee$ for any $k \gg 1$, which follows from [Laz04a, Example 2.2.9]. In particular, this shows that (1) $\implies$ (3).

Let us now show (3) $\implies$ (2). Let $i : V \hookrightarrow X$ be the inclusion of the biggest open subset on which $\mathcal{F}$ is locally free. Given any ample line bundle $\mathcal{H}$, there exists $\gamma > 0$ such that $\widehat{S}^\gamma\mathcal{F} \otimes \mathcal{H}^{-1}$ is weakly positive. Therefore, there exists an integer $\beta > 0$ such that $\widehat{S}^{2\cdot\beta}(\widehat{S}^\gamma\mathcal{F} \otimes \mathcal{H}^{-1}) \otimes_{\mathcal{O}_X} \mathcal{H}^\beta \simeq \widehat{S}^{2\cdot\beta}(\widehat{S}^\gamma\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{H}^{-2\beta+\beta}$ is generically globally generated, as well as the quotient sheaf $\widehat{S}^{2\cdot\beta\cdot\gamma}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}^{-\beta}$. This shows the existence of a generically surjective map $\oplus\mathcal{H}^\beta \to \widehat{S}^{2\cdot\beta\cdot\gamma}\mathcal{F}$. On the other hand, the non-zero map $\mathcal{F}_{|V} \to S^k\mathcal{E}_{|V}$ corresponds to a non-zero map $(\pi^*\mathcal{F})_{|\pi^{-1}(V)} \to \mathcal{O}_{\mathcal{E}}(k)_{|\pi^{-1}(V)}$, which in turn provides a non-zero map $(S^{2\cdot\beta\cdot\gamma}\pi^*\mathcal{F})_{|\pi^{-1}(V)} \to \mathcal{O}_{\mathcal{E}}(2\cdot\beta\cdot\gamma\cdot k)_{|\pi^{-1}(V)}$, or equivalently a non-zero map $S^{2\cdot\beta\cdot\gamma}\mathcal{F}_{|V} \to S^{2\cdot\beta\cdot\gamma\cdot k}\mathcal{E}_{|V}$. Finally, by composing the generically surjective map $\oplus\mathcal{H}^\beta \to \widehat{S}^{2\cdot\beta\cdot\gamma}\mathcal{F}$ with the non-zero map $i_*(S^{2\cdot\beta\cdot\gamma}\mathcal{F}_{|V}) \to i_*(S^{2\cdot\beta\cdot\gamma\cdot k}\mathcal{E}_{|V})$, we get a non-zero map $\oplus\mathcal{H}^\beta \to S^{2\cdot\beta\cdot\gamma\cdot k}\mathcal{E}$, hence a non-zero map $\mathcal{H}^\beta \to S^{2\cdot\beta\cdot\gamma\cdot k}\mathcal{E}$.

Let us finally show (2) $\implies$ (1), following [Laz04b, Example 6.1.23]. Assume that there exists an injective map $0 \to \mathcal{H} \to S^k\mathcal{E}$ for some $k > 0$ (using Kodaira's lemma, cf. [Laz04a, Proposition 2.2.6], one can assume that $\mathcal{H}$ is ample). Equivalently, the line bundle $\mathcal{O}_{\mathcal{E}}(k) \otimes \pi^*\mathcal{H}^{-1}$ has a non-zero section. On the other hand, as $\mathcal{O}_{\mathcal{E}}(1)$ is relatively ample, $\mathcal{O}_{\mathcal{E}}(1) \otimes \pi^*\mathcal{H}$ is ample (cf. [Laz04a, Proposition 1.7.10]). It follows that $\mathcal{O}_{\mathcal{E}}(k+1) = (\mathcal{O}_{\mathcal{E}}(k) \otimes \pi^*\mathcal{H}^{-1}) \otimes \mathcal{O}_{\mathcal{E}}(1) \otimes \pi^*\mathcal{H}$ is big. $\square$

2.3. Complex algebraic varieties with maximal cotangent dimension.

Definition 2.7. A complex algebraic variety X is said to have maximal cotangent dimension if any smooth projective complex variety birational to X has a big cotangent bundle.

If X is a smooth projective complex variety with maximal cotangent dimension, then any smooth projective complex variety birational to X has the same property, cf. [Sak79].

Lemma 2.8. *Let $f : X \to Y$ be a generically finite and dominant algebraic map between two complex algebraic varieties. If Y has maximal cotangent dimension, then the same is true for X.*

Theorem 2.9 (Campana-Peternell, Campana-Păun, [CP15], see also [Cla17, Corollary 2.24] and the references therein)**.** *Any complex algebraic variety of maximal cotangent dimension is of general type.*

3. A Hodge-theoretic criterion for hyperbolicity

In this section we explain a Hodge-theoretic criterion for proving that a variety is of general type. Before stating the criterion, we need to introduce a few definitions.

Definition 3.1. Let V be a complex vector space of finite dimension. A (complex) Hodge structure (of weight zero) on V is a decomposition $V = \bigoplus_{p\in\mathbb{Z}} V^p$. A polarization of a complex Hodge structure is a non-degenerate hermitian form h on V making the decomposition $V = \bigoplus_{p\in\mathbb{Z}} V^p$ orthogonal, such that the restriction of h to V^p is positive definite for p even and negative definite for p odd. The associated Hodge filtration is the decreasing filtration F on V defined by $F^p := \bigoplus_{q\geq p} V^q$.

If V underlies a complex Hodge structure polarized by h, then the Hodge filtration determines the Hodge structure thanks to the formula:

$$V^p = F^p \cap (F^{p+1})^{\perp_h}.$$

Therefore it is equivalent to give the Hodge decomposition of V or the associated Hodge filtration.

Definition 3.2 (Log $\mathbb{C}$-PVHS)**.** Let X be a complex manifold and $D \subset X$ be a normal crossing divisor. A log complex polarized variation of Hodge structure (log $\mathbb{C}$-PVHS) on the log-pair (X, D) consists of the following data:

- A holomorphic vector bundle $\mathcal{V}$ on X endowed with a connection ∇ with logarithmic singularities along D.
- An exhaustive decreasing filtration $\mathcal{F}^\bullet$ indexed by $\mathbb{Z}$ on $\mathcal{V}$ by holomorphic subbundles (the Hodge filtration), satisfying Griffiths transversality
$$\nabla\mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_X(\log D).$$
- a ∇-flat non-degenerate hermitian form h on $\mathcal{V}_{|X-D}$

such that for all $s \in X - D$ the filtered vector space $(\mathcal{V}_s, \mathcal{F}^\bullet_s)$ defines a Hodge structure polarized by h_s.

When $D = \emptyset$, we recover the notion of complex polarized variation of Hodge structure, cf. [Del87, Sim92].

We can now state the main result of this section.

Theorem 3.3. *Let $\mathbb{V} = (\mathcal{V}, \nabla, \mathcal{F}^\bullet, h)$ be a log $\mathbb{C}$-PVHS of length w on a projective smooth log pair $(\bar{X}, D)$ with nilpotent residues along the irreducible components of D. Assume that*

(1) the smallest nonzero Hodge subbundle $\mathcal{F}^{max}$ is a line bundle,
(2) the line bundle $\mathcal{F}^{max}(-w\cdot D)$ is big.

Then the cotangent bundle of $\bar{X}$ is big (hence a fortiori $\bar{X}$ is of general type).

By definition, if $[a, b]$ is the smallest interval such that $\mathrm{Gr}^i_{\mathcal{F}}\mathcal{V} = 0$ for $i \notin [a, b]$, then the length of $\mathbb{V}$ is the integer $b - a$ and $\mathcal{F}^{max} := \mathcal{F}^b$. Note that by renumbering the Hodge filtration, one can always assume that $a = 0$ and $b = w$.

By setting $\mathcal{E} := \mathrm{Gr}_{\mathcal{F}}\mathcal{V}$ and $\theta := \mathrm{Gr}_{\mathcal{F}}\nabla$, we define a system of log Hodge bundles $(\mathcal{E}, \theta)$. It consists of a holomorphic vector bundle $\mathcal{E}$ on $\bar{X}$ with a decomposition $\mathcal{E} = \bigoplus_{p\in\mathbb{Z}} \mathcal{E}^p$ as a sum of holomorphic subvector bundles $\mathcal{E}^p := \mathcal{F}^p/\mathcal{F}^{p+1}$, and a holomorphic 1-form $\theta \in \Omega^1_{\bar{X}}(\log D) \otimes \mathrm{End}(\mathcal{E})$ (the Higgs field) which satisfies $\theta\wedge\theta = 0 \in \Omega^2_{\bar{X}}(\log D) \otimes \mathrm{End}(\mathcal{E})$ and $\theta(\mathcal{E}^p) \subset \mathcal{E}^{p-1} \otimes \Omega^1_{\bar{X}}(\log D)$.

The Higgs field θ corresponds to an $\mathcal{O}_{\bar{X}}$-linear morphism $\phi : T_{\bar{X}}(-\log D) \to \mathrm{End}(\mathcal{E})$. The condition $\theta\wedge\theta = 0$ implies that for every $k \geq 1$, the induced morphism

$T_{\bar{X}}(-\log D)^{\otimes k} \to \mathrm{End}(\mathcal{E})$ factorizes through $\mathrm{Sym}^k T_{\bar{X}}(-\log D)$.

Before giving the proof of Theorem 3.3, let us recall the following weak positivity statement proved by Popa and Wu [PW16, Corollary 3.4] as an easy consequence of previous results of Zuo [Zuo00, Theorem 1.2] and the author [Bru18, Theorem 1.6]. See also [Bru17] for further developments.

Theorem 3.4. *Consider a log $\mathbb{C}$-PVHS on a projective smooth log pair $(\bar{X}, D)$ with nilpotent residues along the irreducible components of D, and denote by $(\mathcal{E}, \theta)$ the corresponding system of log Hodge bundles. If $\mathcal{A}$ is a coherent subsheaf of $\mathcal{E}$ contained in the kernel of the Higgs field θ, then its dual $\mathcal{A}^\vee$ is a weakly positive torsion-free sheaf.*

Proof of Theorem 3.3. With the notations already introduced, and up to renumbering the Hodge filtration, one has $\mathcal{E}^w = \mathcal{F}^{max}$. For every integer $k \geq 0$, the k-iterated Higgs field defines a morphism of $\mathcal{O}_{\bar{X}}$-modules

$$\phi_k : Sym^k T_{\bar{X}}(-\log D) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{E}^w \to \mathcal{E}^{w-k}.$$

Since $\phi_k = 0$ when $k \geq w+1$, there is a largest integer $k \leq w$ for which the map ϕ_k is non-zero. Note that $k \geq 1$: otherwise the line bundle $\mathcal{F}^{max}$ would be contained in the kernel of the Higgs field, hence its dual would be weakly positive by Theorem 3.4. But then the line bundle $\mathcal{O}_{\bar{X}} = \mathcal{F}^{max} \otimes (\mathcal{F}^{max})^\vee$ would be big (by the condition (2) the line bundle $\mathcal{F}^{max}(-w \cdot D)$ is big, hence a fortiori $\mathcal{F}^{max}$ is big too), a contradiction.

Denote by $\mathcal{N}$ the image of ϕ_k. It is a coherent subsheaf of $\mathcal{E}^{w-k}$ contained in the kernel of θ. It follows from Theorem 3.4 that its dual $\mathcal{N}^\vee$ is a weakly positive torsion-free coherent sheaf. From ϕ_k we get a non-zero morphism

$$\mathcal{N}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{E}^w \to Sym^k \Omega^1_{\bar{X}}(\log D).$$

By tensoring with the line bundle $\mathcal{O}_{\bar{X}}(-w \cdot D)$, this provides a non-zero morphism

$$\mathcal{N}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{E}^w(-w \cdot D) \to Sym^k \Omega^1_{\bar{X}}(\log D) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{\bar{X}}(-w \cdot D).$$

Since $k \leq w$, we have an inclusion of sheaves

$$Sym^k \Omega^1_{\bar{X}}(\log D) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{\bar{X}}(-w \cdot D) \subset Sym^k \Omega^1_{\bar{X}},$$

hence finally we obtain a non-zero morphism

$$\mathcal{N}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{E}^w(-w \cdot D) \to Sym^k \Omega^1_{\bar{X}}.$$

Since $\mathcal{N}^\vee$ is weakly positive and $\mathcal{E}^w(-w \cdot D)$ is big, it follows by Lemma 2.4 (iv) that the left-hand side is Viehweg-big, therefore by Lemma 2.5 the vector bundle $\Omega^1_{\bar{X}}$ is big. □

4. Hyperbolicity of Siegel modular varieties

Let g and n be two positive integers. We denote by $\mathcal{A}_g(n)$ the moduli stack of principally polarized complex abelian varieties with a level-n structure, and by $A_g(n)$ the corresponding coarse moduli space. Recall that a level-n structure on a principally polarized abelian variety A of dimension g over a field k of characteristic zero is a $2g$-tuple of points in $A(k)$ which generate the subgroup of n-torsion points in $A(\bar{k})$ and form a symplectic basis with respect to the Weil pairing. From now on we assume that $n \geq 3$, so that $\mathcal{A}_g(n) = A_g(n)$ is a smooth quasi-projective complex variety. Let $\pi : \mathcal{X}_g(n) \to \mathcal{A}_g(n)$ be the universal family of principally polarized abelian varieties with a level-n structure. The complex local system $R^g\pi_*(\underline{\mathbb{C}}_{\mathcal{X}_g(n)})$ underlies a canonical structure of $\mathbb{C}$-PVHS (which in this case has even a canonical $\mathbb{Z}$-structure) that extends canonically to any smooth projective toroidal compactification $\bar{\mathcal{A}}_g(n)$ as a log $\mathbb{C}$-PVHS. More precisely, denoting by D the simple normal crossing boundary divisor $\bar{\mathcal{A}}_g(n) - \mathcal{A}_g(n)$, we obtain a log $\mathbb{C}$-PVHS on $(\bar{\mathcal{A}}_g(n), D)$ as follows. Since $n \geq 3$, the monodromy at infinity of the local system $R^g\pi_*(\underline{\mathbb{C}}_{\mathcal{X}_g(n)})$ is unipotent, hence the associated flat bundle has a canonical Deligne extension $(\mathcal{V}, \nabla)$ with nilpotent residues along the irreducible components of D. Moreover, the nilpotent orbit theorem of Schmid [Sch73] implies that the Hodge filtration extends as a filtration $\mathcal{F}$ of $\mathcal{V}$ by subbundles. We obtain in this way a log $\mathbb{C}$-PVHS on $(\bar{\mathcal{A}}_g(n), D)$ with nilpotent residues along the irreducible components of D. It is well-known that its smallest Hodge subbundle $\mathcal{L} := \mathcal{F}^{max}$ is the canonical extension to $\bar{\mathcal{A}}_g(n)$ of the Hodge line bundle $\pi_*(\omega_{\mathcal{X}_g(n)/\mathcal{A}_g(n)})$ on $\mathcal{A}_g(n)$.

As a direct application of Theorem 3.3, we obtain the following criterion.

Theorem 4.1. *If the line bundle $\mathcal{L}(-g \cdot D)$ on $\bar{\mathcal{A}}_g(n)$ is Viehweg-big over $\mathcal{A}_g(n)$, then all subvarieties of $\mathcal{A}_g(n)$ are of maximal cotangent dimension (hence a fortiori of general type).*

Remark 4.2. Since $\mathcal{L}^{\otimes(g+1)}$ is isomorphic to the log-canonical bundle $\omega_{\bar{\mathcal{A}}_g(n)}(D)$ of $(\bar{\mathcal{A}}_g(n), D)$, the line bundle $\mathcal{L}(-g \cdot D)$ is Viehweg-big over $\mathcal{A}_g(n)$ exactly when the line bundle $\omega_{\bar{\mathcal{A}}_g(n)}((1 - g(g+1)) \cdot D)$ is Viehweg-big over $\mathcal{A}_g(n)$.

Proof of Theorem 4.1. Let Y be a subvariety of $\mathcal{A}_g(n)$ and let $(\bar{Y}, E)$ be a log-compactification of a desingularization of Y. Up to blowing-up E, one can assume that there is a map of log-pairs $f : (\bar{Y}, E) \to (\bar{\mathcal{A}}_g(n), D)$. The $\mathbb{C}$-PVHS constructed above induces by pull-back a $\mathbb{C}$-PVHS of length g on $(\bar{Y}, E)$ whose smallest Hodge subbundle is the pull-back of $\mathcal{L}$. It follows from the hypothesis and Lemma 2.4 (v) that the pull-back along f of the line bundle $\mathcal{L}(-g \cdot D)$ is big. A fortiori the line bundle $(f^*\mathcal{L})(-g \cdot E)$ is big and one is therefore in position to apply the Theorem 3.3. □

If $\bar{\mathcal{A}}_g(n)$ denotes the first Voronoi compactification of $\mathcal{A}_g(n)$ and D the reduced boundary divisor, then Shepherd-Barron [SB06, Theorem 4.1] proved that for any positive rational number a the $\mathbb{Q}$-line bundle $\mathcal{L}(-a \cdot D)$ is ample exactly when $n > 12 \cdot a$. In view of Theorem 4.1, we obtain the following result.

Theorem 4.3. *If $n > 12 \cdot g$, then all subvarieties of $\mathcal{A}_g(n)$ are of maximal cotangent dimension (hence a fortiori of general type).*

5. Hyperbolicity of arithmetic locally symmetric varieties

5.1. **Generalities on bounded symmetric domains and their quotients.** To fix the notations and help the reader which is not familiar with arithmetic locally symmetric varieties, we collect in this section some facts about them that will be used in the sequel. References include [AMRT75, chapter III §2] and [Del71, Del79, Hel78, Mil13, Mok89].

Let $\mathcal{D}$ be a bounded symmetric domain, i.e. a bounded connected open subset in some $\mathbb{C}^n$ such that every $z \in D$ is an isolated fixed point of an holomorphic involution. We denote by $\mathrm{Aut}(\mathcal{D})$ its group of holomorphic automorphisms, and by $\mathrm{Is}(\mathcal{D})$ its group of isometries with respect to the riemannian structure defined by the Bergman metric (cf. [Mok89, chapter 4]). Both $\mathrm{Aut}(\mathcal{D})$ and $\mathrm{Is}(\mathcal{D})$ are semi-simple real Lie groups with finitely many connected components, and $\mathrm{Aut}(\mathcal{D})^+ = \mathrm{Is}(\mathcal{D})^+$ is a semi-simple real Lie group with trivial center. In particular, $\mathrm{Is}(\mathcal{D})^+ = \mathrm{Ad}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of $\mathrm{Is}(\mathcal{D})$ and $^+$ denotes the connected component of the identity in the euclidean topology. (In fact, it is known that $\mathrm{Is}(\mathcal{D}) = \mathrm{Aut}(\mathfrak{g})$.) Let $\mathbf{G}$ be the connected component of the identity subgroup of the real algebraic group $\mathbf{Aut}(\mathfrak{g})$, so that $\mathbf{G}(\mathbb{R})^+ = \mathrm{Aut}(\mathcal{D})^+ \subset \mathbf{Aut}(\mathfrak{g})(\mathbb{R})$. We call $\mathbf{G}$ the real algebraic group associated to $\mathcal{D}$. It is a connected semi-simple real algebraic group of adjoint type. For every $z \in \mathcal{D}$, the subgroup $K_z \subset \mathbf{G}(\mathbb{R})^+$ of biholomorphisms fixing z is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})^+$. Choosing a basepoint z for $\mathcal{D}$, we have an identification $\mathcal{D} = \mathbf{G}(\mathbb{R})^+/K_z$.

We call $\mathcal{D}$ irreducible when $\mathbf{G}$ is simple. In general, $\mathcal{D}$ can be decomposed uniquely as a product of irreducible bounded symmetric domains, and this decomposition corresponds to the decomposition of $\mathbf{G}$ as a direct product of its simple subgroups. The rank of $\mathcal{D}$ is by definition the rank of the real algebraic group $\mathbf{G}$, i.e. the dimension of a maximal split torus. It is denoted by $\mathrm{rk}\,\mathcal{D}$.

A subgroup Γ of $\mathrm{Aut}(\mathcal{D})$ is called arithmetic if there exists a $\mathbb{Q}$-form $\mathbf{G}_{\mathbb{Q}}$ of $\mathbf{G}$ and an embedding $\mathbf{G}_{\mathbb{Q}} \hookrightarrow \mathbf{Gl}_n$ defined over $\mathbb{Q}$ such that Γ is commensurable with $\mathbf{G}_{\mathbb{Q}}(\mathbb{Q}) \cap \mathbf{Gl}_n(\mathbb{Z})$, i.e. the intersection is of finite index in each. This property turns out to be independent of the embedding. Given a $\mathbb{Q}$-group $\mathbf{G}_{\mathbb{Q}}$, a subgroup $\Gamma \subset \mathbf{G}_{\mathbb{Q}}(\mathbb{Q})$ is called neat (cf. [Bor69, §17.1]) if for any representation $\mathbf{G}_{\mathbb{Q}} \to \mathbf{Gl}_n$ defined over $\mathbb{Q}$ and any element $\gamma \in \Gamma$, the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of the automorphism of $\mathbb{C}^n$ associated to γ is torsion-free. In particular, Γ is torsion-free. Moreover, any arithmetic group admits a finite-index neat subgroup.

Definition 5.1. An arithmetic locally symmetric variety is a complex analytic space which is isomorphic to a quotient of a bounded symmetric domain $\mathcal{D}$ by an arithmetic subgroup $\Gamma \subset \mathrm{Aut}(\mathcal{D})$.

It follows from the work of Baily-Borel [BB66] that every arithmetic locally symmetric variety $X = \Gamma \backslash \mathcal{D}$ admits a canonical compactification by a normal projective variety that we denote $\overline{X}^*$. In particular, X admits a canonical structure of quasi-projective variety. However, Igusa and others have shown that the singularities of $\overline{X}^*$ are very complicated in general. In order to address this problem, Mumford et al. [AMRT75] have introduced the collection of so-called toroidal compactifications of X, which are algebraic spaces with only quotient singularities. These compactifications are not unique, they depend on the choice of a combinatorial data Σ. Moreover, when Γ is neat, it is always possible to find a toroidal compactification $\overline{X}^{\Sigma}$ of X such that $(\overline{X}^{\Sigma}, D)$ is a smooth projective log-compactification of X, where $D := \overline{X}^{\Sigma} - X$. For any toroidal compactification $\overline{X}^{\Sigma}$, the identity map $X \to X$ can always be extended to a holomorphic map $\overline{X}^{\Sigma} \to \overline{X}^*$.

When $\mathcal{D}$ is irreducible, the group of isomorphism classes of $\mathbf{G}(\mathbb{R})^+$-equivariant holomorphic line bundles on $\mathcal{D}$ is infinite cyclic. We denote by $\mathcal{L}$ the generator such that the canonical bundle of $\mathcal{D}$ is a positive power of $\mathcal{L}$. When $\mathcal{D}$ is reducible, one defines $\mathcal{L}$ to be the tensor product of the $\mathbf{G}(\mathbb{R})^+$-equivariant line bundles obtained as before. For any arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$, the holomorphic line bundle $\mathcal{L}$ descends to the arithmetic locally symmetric variety $X = \Gamma \backslash \mathcal{D}$ as a $\mathbb{Q}$-line bundle (an honnest line bundle when Γ is torsion-free), and has a canonical extension to any toroidal compactification $\bar{X}$ that we still denote by $\mathcal{L}$. The obtained $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\bar{X}$ is called the automorphic line bundle, and it is well-known that it is Viehweg-big over X (in fact sections of high powers of $\mathcal{L}$ define the canonical map from $\bar{X}$ to the Baily-Borel compactification of X).

5.2. **Proof of Theorem 1.1.** Let $\mathcal{D}$ be a bounded symmetric domain and $\mathbf{G}$ be the corresponding real algebraic group of adjoint type. Gross [Gro94] (in the tube domain case) and Sheng-Zuo [SZ10] (in general) have constructed a $\mathbf{G}(\mathbb{R})^+$-equivariant $\mathbb{C}$-PVHS which, among other properties, is effective of weight $n = \operatorname{rk} \mathcal{D}$ and whose smallest Hodge bundle can be identified with the automorphic line bundle. This so-called automorphic $\mathbb{C}$-PVHS of Calabi-Yau type induces in turn a $\mathbb{C}$-PVHS on any quotient $\Gamma \backslash \mathcal{D}$ of $\mathcal{D}$ by a torsion-free discrete subgroup $\Gamma \subset \mathbf{G}(\mathbb{R})^+$. (In the case of the Siegel modular variety, it turns out that we recover by this group-theoretic construction the $\mathbb{C}$-PVHS considered in section 4.) An easy adaptation of the arguments of section 4 yields the following criterion.

Theorem 5.2. *Let $X = \Gamma \backslash \mathcal{D}$ be a quotient of a bounded symmetric domain $\mathcal{D}$ by a neat arithmetic lattice $\Gamma \subset \mathbf{G}(\mathbb{R})^+$. Let $r_{\mathcal{D}} = \operatorname{rk} \mathcal{D}$ be the rank of $\mathcal{D}$. Assume that the following condition $(*)$ is satisfied: for some smooth projective toroidal compactification $\bar{X}$ of X with boundary divisor D and automorphic line bundle $\mathcal{L}$, the line bundle $\mathcal{L}(-r_{\mathcal{D}} \cdot D)$ is Viehweg-big over X.*
Then all subvarieties of X are of maximal cotangent dimension (hence a fortiori of general type).

It is easily seen that the condition $(*)$ is independent of the chosen compactification. Moreover, if Γ satisfies $(*)$, then any finite index subgroup of Γ satisfies $(*)$ too. To prove Theorem 1.1, we will now explain that $(*)$ is always satisfied when Γ is small enough.

Fix a $\mathbb{Q}$-form $\mathbf{G}_{\mathbb{Q}}$ of $\mathbf{G}$ and an embedding $\mathbf{G}_{\mathbb{Q}} \hookrightarrow \mathbf{Gl}_m$ defined over $\mathbb{Q}$, and consider the arithmetic locally symmetric variety X associated to the arithmetic group $\mathbf{G}_{\mathbb{Q}}(\mathbb{Q}) \cap \mathbf{Gl}_m(\mathbb{Z})$. For any positive integer n, we have the finite cover $\pi_n : X(n) \to X$ associated to the principal congruence subgroup

$$\Gamma(n) := \mathbf{G}_{\mathbb{Q}}(\mathbb{Q}) \cap \ker(\mathbf{Gl}_m(\mathbb{Z}) \to \mathbf{Gl}_m(\mathbb{Z}/n\mathbb{Z})).$$

For any choice of combinatorial data Σ, the map $\pi_n : X(n) \to X$ extends canonically to the corresponding toroidal compactifications $\pi_n : \bar{X}(n) \to \bar{X}$. Since the $\mathbb{Q}$-line bundle $\mathcal{L}$ is Viehweg-big over X, the same is true for the $\mathbb{Q}$-line bundle $\mathcal{L}(-\epsilon \cdot D)$ for any $\epsilon > 0$ small enough. Applying Lemma 2.4 (v), we infers that the $\mathbb{Q}$-line bundle $\pi_n^*(\mathcal{L}(-\epsilon \cdot D))$ is Viehweg-big over $X(n)$. Since the map π_n is highly ramified over D [Mum77, pp. 269-272], it follows that the condition $(*)$ is satisfied for n big enough. Therefore we have proved:

Theorem 5.3. *Let $\mathcal{D}$ be a bounded symmetric domain and $\mathbf{G}$ be the corresponding real algebraic group of adjoint type. Fix a $\mathbb{Q}$-form $\mathbf{G}_{\mathbb{Q}}$ of $\mathbf{G}$ and an embedding $\mathbf{G}_{\mathbb{Q}} \hookrightarrow \mathbf{Gl}_m$ defined over $\mathbb{Q}$, and denote by $\Gamma(n)$ the associated sequence of principal congruence subgroups. Then, for n big enough, all subvarieties of $\Gamma(n)\backslash\mathcal{D}$ are of maximal cotangent dimension (hence a fortiori of general type).*

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